# Notes on Complex Numbers
## Math 170: Ideas in Mathematics (Section 002)

University of Pennsylvania. October 7, 2014.
Instructor: Subhrajit Bhattacharya

## Imaginary Numbers

The set of real algebraic numbers, $\mathbb{A}$, consists of the real numbers that are solution to polynomial equations with rational coefficients.
**Examples:**

i. $\sqrt{2} \in \mathbb{A}$ since it is a solution to the polynomial equation $x^2 - 2 = 0$.

ii. $-2 + \sqrt{3} \in \mathbb{A}$ since it is a solution to the polynomial equation $x^2 + 4x + 1 = 0$.

However, there are many polynomial equations whose solutions cannot be found anywhere in the set of real numbers. For example, consider the simple polynomial equation $x^2 + 2 = 0$. A solution to this, $x$, should be such that $x^2 = -2$. However the square of any real number cannot be negative. So we introduce new numbers in our number system:

> We take the simplest of such polynomial equations, $x^2 + 1 = 0$, and for a solution to this equation define a new number: $i = \sqrt{-1}$.

This new number, $i$, that we just defined is called *imaginary* due to historic reasons. However, in the light of modern mathematics this is no more "imaginary" than the rational numbers, real algebraic numbers or the transcendental numbers — each of which were introduced in our number system in order to *complete* some structure, operation or properties on the number system. For example, when we introduced the rationals, we saw that some ratio of integers, such as $\frac{6}{3}, \frac{-21}{7}$, etc. do lie in the set of integers themselves. But the ratios like $\frac{3}{2}, \frac{7}{9}, \frac{20}{6}$, etc do not find a place in $\mathbb{Z}$. So we introduced new numbers in the system – every number of the form $\frac{p}{q}$ for $p, q \in \mathbb{Z}$ – and called this new set of numbers *rational numbers*, $\mathbb{Q}$. Likewise, now we introduce new numbers for solution to polynomial equations that clearly do not have solution on the real line, $\mathbb{R}$.

Since it is clear that this number, $i$, does not lie on $\mathbb{R}$, we place it somewhere outside the number line. It is also clear that if a solution to the equation of $x^2 + 1 = 0$ is written as $i$, then there will be polynomial equations whose solutions will be multiples of $i$. For example, for the polynomial equation $x^2 + 4 = 0$, a solution is $x = \sqrt{-4} = \sqrt{4}\sqrt{-1} = 2i$. As another example, a solution to $x^2 + \frac{2}{9}$ is $x = -\sqrt{-\frac{2}{9}} =$

$-\frac{\sqrt{2}}{3}i$. These multiples of *i* are called "*imaginary numbers*", although as mentioned earlier, there is absolutely nothing "imaginary" about them. It's just a name given to these numbers.
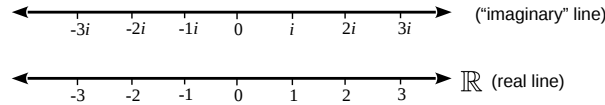


Figure 1: The real and imaginary number lines.

Thus, now we have two number lines in our number system: one is our usual real number line, $\mathbb{R}$, and the other is the *imaginary line* containing the multiples of *i* (Figure 1).

## Complex numbers

However, the next thing that we observe is that the solutions to some polynomial equations come in form of a sum of a real number and an imaginary number. For example, consider the polynomial equation $x^2 - 2x + 3 = 0$, which can be re-written as $(x - 1)^2 + 2 = 0$. Thus, a solution to this equation is given by $x - 1 = \sqrt{-2} = \sqrt{2}i \Rightarrow x = 1 + \sqrt{2}i$. Thus, the number $1 + \sqrt{2}i$ is a solution to a polynomial equation. In general, one can get arbitrary numbers of the form $a + bi$, where $a, b \in \mathbb{R}$, as solutions to polynomial equations. These numbers are essentially made up of two real numbers, *a* and *b* (the later being the coefficient of *i*), and are called *complex numbers*.

We emphasize once again that complex numbers are numbers that are made up of nothing but pairs of real numbers, $(a, b)$, written in the peculiar form $a + bi$. Thus complex numbers an be represented as points on a plane, which by definition, is $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, consisting pairs of real numbers as its elements (recall the definition of a Cartesian product). This plane of the complex numbers is called the *complex plane*, $\mathbb{C}$, and just as we mentioned, this is nothing but $\mathbb{R}^2$. So the complex number $a + bi$ can be drawn as point on the complex plane with a projection of *a* on the horizontal axis (called the *real axis*) and a projection of *b* on the vertical axis (called the *imaginary* axis). This is illustrated in Figure 2.

### The Fundamental Theorem of Algebra

So far we have defined $i = \sqrt{-1}$ (a solution to the polynomial equation $x^2 + 1 = 0$) and saw that solution to some polynomial equations can be written in the form of
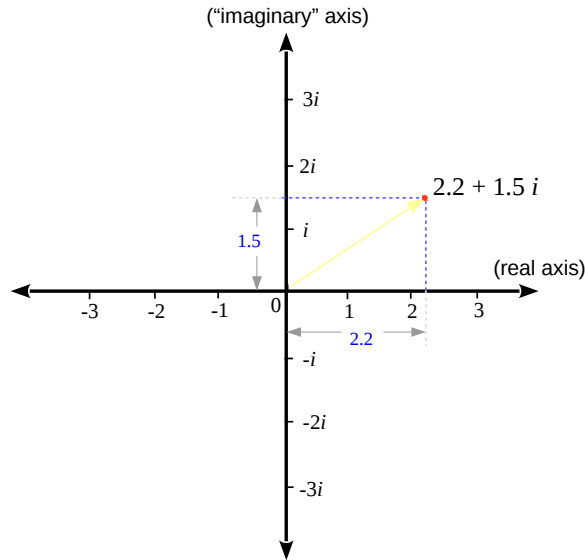
Figure 2: The complex plane.

$a + ib$, where $(a, b) \in \mathbb{R}^2$. But what is so special about $\sqrt{-1}$? What about the polynomial equation $x^4 + 1 = 0$? A solution to this equation seems to be $\sqrt[4]{-1}$. Do we have to define another new number, say $j = \sqrt[4]{-1}$, for a solution to this polynomial?

It turns out that we need not define any more new numbers for writing solution to polynomial equations with any real coefficients. It can in fact be shown that a solution to the aforesaid equation, $x^4 + 1 = 0$, is actually $x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ (a complex number). We can verify that very easily: If $x = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, then,

$$x^2 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}i\right) + \left(\frac{1}{\sqrt{2}}i\right)^2 = \frac{1}{2} + 2\frac{1}{2}i - \frac{1}{2} = i$$

Thus we have, $x^4 = (x^2)^2 = i^2 = -1$. This means $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ is a solution to the polynomial equation $x^4 + 1 = 0$. In fact any polynomial equation with real or complex coefficients have solutions that are complex numbers (*i.e.* are of the form $a + bi$). This is formally known under the name of *The Fundamental Theorem of Algebra*.

> **The Fundamental Theorem of Algebra:** Every polynomial equation in $x$, with real or complex coefficients, have one or more solutions which are complex numbers.

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## Complex Algebra

In this section we will describe the rules for addition, multiplication and division of complex numbers:

*Addition:* $(a+bi)+(p+qi)=(a+p)+(b+q)i.$
Example: $(2+3i)+(1-1.4i)=3+1.6i.$

*Subtraction:* $(a+bi)-(p+qi)=(a-p)+(b-q)i.$
Example: $(2+3i)-(1-1.4i)=1+4.4i.$

*Multiplication:* $(a+bi)(p+qi)=ap+bpi+aqi+(bi)(qi)=(ap-bq)+(bp+aq)i.$
Example: $(2+3i)(1-1.4i)=6.2+0.2i.$

*Multiplication:* $\frac{a+bi}{p+qi}=\frac{(a+bi)(p-qi)}{(p+qi)(p-qi)}=\frac{(ap+bq)+(bp-aq)i}{p^2+q^2}.$
Example: $\frac{2+3i}{1-2i}=\frac{-5+7i}{5}=-1+\frac{7}{5}i.$
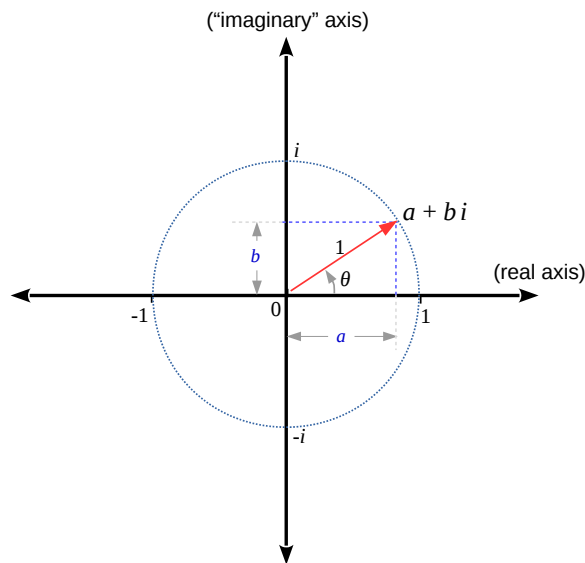
## Euler's Formula



Figure 3: The unit circle in the complex plane.

Consider the circle centered at the origin in the complex plane and having a radius of 1 (Figure 2). This is called the *unit circle*. A point $a+bi$ (a complex number) lying on that circle will be at a distance of 1 from the origin. Looking at

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the right angle triangle with the base angle $\theta$ as shown in Figure 2, one can use trigonometry to immediately observe that $a = \cos(\theta)$ and $b = \sin(\theta)$. This lets us write $a + bi = \cos(\theta) + i\sin(\theta)$. The Euler's formula gives another description of this number: $e^{i\theta}$.

**Euler's Formula:** $e^{i\theta} = \cos(\theta) + i\sin(\theta)$

where, $e = 2.71828\ldots$ is an irrational (and in fact transcendental) number, and $e^x$ is called the *exponential function*. While we may know what a real number raised to the power of another real number is (*e.g.*, $e^2 = e \cdot e$, $e^{0.5} = \sqrt{e}$, $e^{-2/3} = \frac{1}{e^{2/3}} = \frac{1}{\sqrt[3]{e^2}}$, etc), it is not obvious what $e$, which is a real constant, raised to the power of an imaginary number, $i\theta$, is. Although we will not discuss the exact meaning/interpretation of imaginary or complex powers, one can take the Euler's formula as definition of $e^{i\theta}$. It matches nicely with the usual properties of and operations on exponentials.

For example, using Euler's formula $e^{i\theta_1}e^{i\theta_2} = (\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2))$. Again, from the properties of exponentials, $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1 + \theta_2)} = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$ (the last Equality is written using Euler's formula). Thus we have $(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) + i\sin(\theta_2)) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$, and this can be verified to be true once you work out the complex multiplication on the left and use some trigonometric identities.

We will end with a few observations at the different value of $\theta$ in the Euler's formula:

i. At $\theta = 0$, the corresponding complex number is $\cos(0) + i\sin(0) = 1$. And indeed $e^{0i} = e^0 = 1$.

ii. At $\theta = \frac{\pi}{2}$ we get $e^{i\frac{\pi}{2}} = i$.

iii. At $\theta = \pi$ we get $e^{i\pi} = -1$. This can be rewritten as $1 + e^{i\pi} = 0$. This is an interesting result since it relates a pure real number, 1, with a pure imaginary number, $i\pi$.

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